

D3/D7 holographic Gauge theory and Chemical potential

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Abstract

N=2 supersymmetric Yang-Mills theory with flavor hypermultiplets at finite temperature and in the dS_4 are studied for finite quark number density (n_b) by a dual supergravity background with non-trivial dilaton and axion. The quarks and its number density n_b are introduced by embedding a probe D7 brane. We find a critical value of the chemical potential at the limit of $n_b = 0$, and it coincides with the effective quark mass given in each theory for $n_b = 0$. At this point, a transition of the D7 embedding configurations occurs between their two typical ones. The phase diagrams of this transition are shown in the plane of chemical potential versus temperature and cosmological constant for YM theory at finite temperature and in dS_4 respectively. In this phase transition, the order parameter is considered as n_b . This result seems to be reasonable since both theories are in the quark deconfinement phase.

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1 Introduction

In the context of the holography [1, 2], the dynamical properties of flavor quarks have been studied by embedding D7 brane(s) as probe. [3, 4, 5, 6, 7, 8, 9, 10, 11]. Recently, the research in this direction has been extended by introducing quark number density. Such researches have been performed at finite temperature for D3/D7 [13, 14, 15] and D4/D8 [12] models, and some kinds of phase transitions are reported.

In D3/D7 model, the high temperature gauge theory is represented by the AdS Schwartzchild configuration, which describes chiral symmetric* and deconfinement phase for zero quark number (n_b). A horizon appears in the bulk configuration in this model, then two types of D7 embeddings are possible, the “Minkowski embedding”, in which the D7 brane is off the horizon, and “black hole embedding” (BH), where the IR end point of the D7 brane touches on the horizon.

For the case of non-zero n_b , however, the authors of [14] have pointed out that the Minkowski embedding is unphysical due to the fact that the inner electric field remains finite at the IR end point of D7 brane, and it can not go nowhere from that point without any appropriate source like a baryon. Therefore, the physical embedding is restricted to the BH type, and they found that all the quark mass range could be covered by the BH embedding. This implies that the BH embedding is enough for finite n_b case.

However, we show here that a finite region of chemical potential ($0 < \mu < \mu_0$) can not be covered by finite n_b and BH embedding, and this region is described rather by the physical Minkowski embedding with $n_b = 0$. And the critical point $\mu_0(= \mu_0(T))$, which depends on the temperature T , is shown to be equivalent to the effective quark mass ($\tilde{m}_q(= \tilde{m}_q(T))$). Then n_b appears when μ exceeds \tilde{m}_q . This result implies that n_b describes the number of quarks moving freely with the mass \tilde{m}_q in the thermal YM medium. This interpretation is very reasonable since the chemical potential is examined here for AdS Schwartzchild background which is dual to the gauge theory in the high temperature deconfinement phase. Then it would be an open problem here how n_b could be related to the baryon number density of a physical baryon. And the curve $\mu_0(T)$ in the μ - T plane indicates the critical line which separates two phases, Minkowski embedding with $n_b = 0$ and BH embedding with $n_b \neq 0$. In passing this critical curve, the chemical potential does not jump but the electric potential and profile function of D7 embedding show a jump near the horizon. The order parameter of this transition is interpreted as $n_b \propto \langle \Psi^\dagger \Psi \rangle$, where Ψ denotes the quark field.

Similar situation to the finite temperature gauge theory is seen also for the gauge theory in the de Sitter space time (dS_4) [21, 22]. This theory is characterized by the 4d cosmological constant (λ), and the configuration of the gravity dual has a horizon as in the high temperature theory. Then both the “Minkowski” and BH D7 brane embeddings are possible. The dynamical role of λ is similar to the temperature so far. In fact, we find for finite λ that the chiral symmetry is restored and the quarks are deconfined. Then it is useful to study the meaning of the “baryon number” n_b also in the dS_4 gauge theory. Actually, we find a similar critical curve $\mu_0(\lambda)$ in the

* Here and hereafter we use the chiral symmetry for the state with massless quark or a hypermultiplet, and the breaking of this symmetry means the spontaneous mass generation for the quark.

μ - λ plane and the same kind of phase transition on this critical curve. We also show $\mu_0(\lambda) = \tilde{m}_q(\lambda)$,

where $\tilde{m}_q(\lambda)$ represents the effective mass of a quark moving freely in dS_4 space.

Thus throughout the gauge theory at finite temperature and in dS_4 , we find that the chemical potential remains finite as μ_0 in the limit of $n_b = 0$ in a definite region of T or λ . In other words, there is an area of $n_b = 0$ and finite $\mu(< \mu_0)$ in μ - T and μ - λ planes. In this area, the D7 brane is embedded by Minkowski type, but the finite chemical potential plays no dynamical role since it disappears in the brane action since the electric potential is given by a constant μ . However, when it exceeds a critical value μ_0 , $n_b(\propto \langle \Psi^\dagger \Psi \rangle)$ appears as a non-zero value and the D7 profile is changed to BH type. This phenomenon could be interpreted as a phase transition with the order parameter $\langle \Psi^\dagger \Psi \rangle$. Our purpose is to give such phase diagrams and to discuss this transition observed in the deconfinement phase.

In section 2, we give the setting of the bulk background and the D7 brane action and the equations of motion of embedding are given in section 3. In the section 4, the chemical potential and its relation to the phase transition is given, and phase diagrams are given in the section 5 and 6. The summary is given in the final section.

2 Background geometry

We start from 10d IIB model retaining the dilaton Φ , axion χ and selfdual five form field strength $F_{(5)}$. Under the Freund-Rubin ansatz for $F_{(5)}$, $F_{\mu_1 \dots \mu_5} = -\sqrt{\Lambda}/2 \epsilon_{\mu_1 \dots \mu_5}$ [16, 17], and for the 10d metric as $M_5 \times S^5$ or $ds^2 = g_{MN} dx^M dx^N + g_{ij} dx^i dx^j$, we find the solution. The five dimensional M_5 part of the solution is obtained by solving the following reduced 5d action,

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left(R + 3\Lambda - \frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}e^{2\Phi}(\partial\chi)^2 \right), \quad (1)$$

which is written in the string frame and taking $\alpha' = g_s = 1$.

Two deconfinement solutions

The following solutions are obtained under the ansatz,

$$\chi = -e^{-\Phi} + \chi_0, \quad (2)$$

which is necessary to obtain a supersymmetric solutions. However, the supersymmetry is broken in the following two solutions due to the cosmological constant λ and the temperature T .

(2) High temperature phase: First, we consider the high temperature gauge theory given in [20], and it is written as

$$\begin{aligned} ds_{10}^2 &= G_{MN} dX^M dX^N \\ &= e^{\Phi/2} \left\{ \frac{r^2}{R^2} \left(-f^2(r) dt^2 + (dx^i)^2 \right) + \frac{1}{f^2(r)} \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2 \right\}, \end{aligned} \quad (3)$$

$$e^\Phi = \left(1 + \frac{q}{r_T^4} \log\left(\frac{1}{1 - (r_T/r)^4}\right)\right), \quad (4)$$

$$f(r) = \sqrt{1 - \left(\frac{r_T}{r}\right)^4}, \quad (5)$$

where where $M, N = 0 \sim 9$ and $R = \sqrt{\Lambda}/2 = (4\pi N)^{1/4}$. The temperature T is related to the parameter r_T as $r_T = \pi R^2 T$, and q represent the VEV of gauge fields condensate [10], $q \propto \langle F_{\mu\nu}^2 \rangle$.

(2) dS₄: The second solution is for the dS₄ solution [22],

$$ds_{10}^2 = e^{\Phi/2} \left\{ \frac{r^2}{R^2} A^2 \left(-dt^2 + a(t)^2 (dx^i)^2 \right) + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2 \right\}, \quad (6)$$

$$e^\Phi = 1 + \frac{q}{r^4} \frac{1 - (r_0/r)^2/3}{(1 - (r_0/r)^2)^3}, \quad (7)$$

$$A = 1 - \left(\frac{r_0}{r}\right)^2, \quad a(t) = e^{2\frac{r_0}{R^2}t} \quad (8)$$

The horizon is denoted by r_0 , which is related to the 4d cosmological constant λ as

$$\lambda = 4 \frac{r_0^2}{R^4}. \quad (9)$$

In this configuration, the four dimensional boundary represents the $\mathcal{N}=4$ SYM theory in the de Sitter space or in the inflational universe characterized by the 4d cosmological constant λ .

3 D7 brane action and equation of motion:

The embedding of the brane is performed by solving the equations of motion for the fields on the brane. The brane action for the D7-probe is given as

$$S_{D7} = -\tau_7 \int d^8\xi \left(e^{-\Phi} \sqrt{-\det(\mathcal{G}_{ab} + 2\pi\alpha' F_{ab})} - \frac{1}{8!} \epsilon^{i_1 \dots i_8} A_{i_1 \dots i_8} \right) \\ + \frac{(2\pi\alpha')^2}{2} \tau_7 \int P[C^{(4)}] \wedge F \wedge F, \quad (10)$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$. $\mathcal{G}_{ab} = \partial_{\xi^a} X^M \partial_{\xi^b} X^N G_{MN}$ ($a, b = 0 \sim 7$, and $M, N = 0 \sim 9$) and $\tau_7 = [(2\pi)^7 g_s \alpha'^4]^{-1}$ represent the induced metric and the tension of D7 brane respectively. And $P[C^{(4)}]$ denotes the pullback of a bulk four form potential,

$$C^{(4)} = \left(\frac{r^4}{R^4} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \right). \quad (11)$$

The eight form potential $A_{i_1 \dots i_8}$, which is the Hodge dual to the axion, couples to the D7 brane minimally. In terms of the Hodge dual field strength, $F_{(9)} = dA_{(8)}$ [18],

the potential $A_{(8)}$ is obtained. In this brane action, there are two scalar fields, which are denoted here as X^8 and X^9 , corresponding to the two coordinates transverse to the brane. Since the brane action is rotational symmetric in the plane of these two coordinate, the embedding is given by the solution of one of the scalar fields, and it is denoted by $w(=X^8)$ given below.

In addition, in the present case, we must solve for the Wick rotated time-component (A_0) of the $U(1)$ vector fields on the brane in order to introduce the chemical potential (μ) of the quark.

For finite temperature :

In the case of finite temperature, the D7 brane is embedded as follows [20]. The six dimensional part of the above metric is rewritten as,

$$\frac{1}{f^2(r)} \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2 = \frac{R^2}{U^2} \left(d\rho^2 + \rho^2 d\Omega_3^2 + (dX^8)^2 + (dX^9)^2 \right), \quad (12)$$

$$U(r) = \exp \left(\int \frac{dr}{r \sqrt{1 - (r_T/r)^4}} \right) = r \sqrt{\frac{1 + f(r)}{2}}. \quad (13)$$

Here U is normalized as $U = r$ for $r_T = 0$, and $U^2 = \rho^2 + (X^8)^2 + (X^9)^2$. Then we obtain the induced metric for D7 brane,

$$ds_8^2 = e^{\Phi/2} \left\{ \frac{r^2}{R^2} \left(-f^2(r) dt^2 + (dx^i)^2 \right) + \frac{R^2}{U^2} \left((1 + (\partial_\rho w^8)^2 + (\partial_\rho w^9)^2) d\rho^2 + \rho^2 d\Omega_3^2 \right) \right\}, \quad (14)$$

where $w^8(\rho)$ and $w^9(\rho)$ are the scalars which determine the position of D7 brane. They are solved under the ansatz that they depend on only ρ . Further we can set $w^9 = 0$ and $w^8 = w(\rho)$ without loss of generality due to the rotational invariance in $X^8 - X^9$ plane. By substituting the above metric in the DBI action and retaining $A_0(\rho)$, we obtain

$$S_{D7}^T = -\tau_7 \int d^8 \xi \sqrt{\epsilon_3} \rho^3 \left(\left(\frac{r}{U} \right)^4 f e^\Phi \sqrt{1 + (w')^2 - \frac{U^2 \tilde{F}_{\rho t, T}^2}{f^2 r^2 e^\Phi}} - \frac{q}{U^4} \right), \quad (15)$$

where $\tilde{F}_{\rho t} = 2\pi\alpha' F_{\rho t}$, $F_{\rho t} = \partial_\rho A_0$ and the last term in the parenthesis denotes the eight form part.

The equations of motion are solved as follows. First, we solve the equation of motion for A_0 by introducing the quark number density n_b as an integral constant as follows,

$$\frac{\rho^3 (r/U)^2 \tilde{F}_{\rho t, T}}{f \sqrt{1 + (w')^2 - \frac{\tilde{F}_{\rho t, T}^2}{f^2 (r/U)^2 e^\Phi}}} = n_b \quad (16)$$

From this, we obtain the following form of $U(1)$ electric field,

$$\tilde{F}_{\rho t, T} = n_b f \frac{r}{U} e^{\Phi/2} \sqrt{\frac{1 + (w')^2}{n_b^2 + \rho^6 (r/U)^6 e^\Phi}}, \quad (17)$$

Here we see the property of $\tilde{F}_{\rho t, T}$ near the horizon $r = r_T$. At this limit, we have

$$\tilde{F}_{\rho t, T} \sim n_b \left(1 - \frac{r_T}{r}\right) \quad (18)$$

for both $q = 0$ and $q \neq 0$. Then $\tilde{F}_{\rho t, T} = 0$ at the horizon, while it is finite at $\rho = 0$ for $n_b \neq 0$. So the only the BH embedding is physical for $n_b \neq 0$, and the Minkowski embeddings are considered only for $n_b = 0$.

Finally, then, the equation of motion for w is given by

$$\begin{aligned} & \frac{w''}{1 + w'^2} + 3 \frac{w'}{\rho} - \frac{w - \rho w'}{U^2 f} \left[2(1 - f)^2 - \frac{4q}{r^4} e^{-\Phi} \right] \\ & - \frac{w(1 + w'^2)^{1/2}}{U^2 f} \frac{4q}{r^4} e^{-\Phi} (1 - G_T)^{1/2} \\ & - G_T \left[\frac{3w'}{\rho} - \frac{(w' \rho - w)}{U^2 f} \left(3(1 - f)f + \frac{2q}{r^4} e^{-\Phi} \right) \right] = 0, \end{aligned} \quad (19)$$

where G_T is defined as

$$G_T = \frac{n_b^2}{n_b^2 + \rho^6 (r/U)^6 e^\Phi}. \quad (20)$$

Of course, this equation reduces to the one given in [20] when $n_b \rightarrow 0$.

For dS_4 :

In this case, the extra six dimensional part of the above metric (6) is rewritten as,

$$\frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2 = \frac{R^2}{r^2} \left(d\rho^2 + \rho^2 d\Omega_3^2 + (dX^8)^2 + (dX^9)^2 \right), \quad (21)$$

where $r^2 = \rho^2 + (X^8)^2 + (X^9)^2$. And we obtain the induced metric for D7 brane,

$$\begin{aligned} ds_8^2 = e^{\Phi/2} & \left\{ \frac{r^2}{R^2} A^2 \left(-dt^2 + a(t)^2 (dx^i)^2 \right) + \right. \\ & \left. \frac{R^2}{r^2} \left((1 + (\partial_\rho w)^2) d\rho^2 + \rho^2 d\Omega_3^2 \right) \right\}, \end{aligned} \quad (22)$$

where we set as $X^9 = 0$ and $X^8 = w(\rho)$ as mentioned above. Then retaining the profile function $w(\rho)$ and $A_0(\rho)$, we arrive at the following D7 brane action,

$$S_{D7} = -\tau_7 \int d^8 \xi \sqrt{\epsilon_3} \rho^3 a(t)^3 \left(A^4 e^\Phi \sqrt{1 + (w')^2 - \frac{\tilde{F}_{\rho t}^2}{A^2 e^\Phi}} - C_8 \right), \quad (23)$$

where $\tilde{F}_{\rho t} = 2\pi\alpha' F_{\rho t}$, $F_{\rho t} = \partial_\rho A_0$ and the eight form part is given as

$$C_8(r) = \int^r dr' A^4(r') \partial_{r'} (\exp(\Phi(r'))) = \frac{q}{r^4}. \quad (24)$$

The equations of motion are solved as above. First, we solve the equation of motion for A_0 by introducing the quark number density n_b as an integral constant,

$$\frac{\rho^3 A^2 \tilde{F}_{\rho t}}{\sqrt{1 + (w')^2 - \frac{\tilde{F}_{\rho t}^2}{A^2 e^\Phi}}} = n_b \quad (25)$$

From this, we obtain the following form of $U(1)$ electric field,

$$\tilde{F}_{\rho t} = n_b A e^{\Phi/2} \sqrt{\frac{1 + (w')^2}{n_b^2 + \rho^6 A^6 e^\Phi}}, \quad (26)$$

Then we get the equation of w by substituting this into the one obtained from D7 action by the variational principle. The final form is written as,

$$\begin{aligned} & \frac{1}{\sqrt{1 + w'^2}} \left\{ \frac{w''}{1 + w'^2} + w' \left[\frac{3}{\rho} + (\Phi + 4 \log A)' - G \left(\frac{3}{\rho} + (3 \log A + \frac{\Phi}{2})' \right) \right] \right\} \\ & + \frac{w}{\rho + w w'} \left\{ \sqrt{1 - G \Phi'} - \sqrt{1 + w'^2} \left[(\Phi + 4 \log A)' - G \left(\frac{\Phi}{2} + 3 \log A \right)' \right] \right\} \\ & = 0, \end{aligned} \quad (27)$$

where prime denotes the derivative with respect to ρ , and G is defined as

$$G = \frac{n_b^2}{n_b^2 + \rho^6 A^6 e^\Phi}. \quad (28)$$

When we take the limit $n_b \rightarrow 0$ at non-zero ρ , the above equation (27) reduces to the one given in [22]. By giving n_b as a parameter, we can solve this equation and find the profile of the embedded D7 brane. Then we find $\tilde{F}_{\rho t}$ simultaneously through (26).

4 Chemical potential and phase transition

From the equations given above, we firstly solve for w with the boundary condition, $w(\infty) = 2\pi\alpha' m_q$, where m_q denotes the current quark mass and we set as $2\pi\alpha' = 1$ hereafter. Then we can read from the asymptotic form of this solution the chiral condensate $\langle \bar{\Psi} \Psi \rangle$, where Ψ denotes the quark field. Then we solve for the gauge potential $A_0(\rho)$ by using this solved w .

In this case, we can assume its asymptotic form at large ρ as in the case of w . In the context of AdS/CFT, it would be written in terms of the chemical potential μ and the quark (or baryon) number density n_b as,

$$A_0(\rho) = \mu - \frac{n_b}{2\rho^2} + \dots, \quad (29)$$

We notice

$$n_b \propto \langle \Psi^\dagger \Psi \rangle = \langle \bar{\Psi} \gamma^0 \Psi \rangle. \quad (30)$$

This would represent the baryon number density when the theory is in the confinement phase, but we call this as quark number here since the present model describes the deconfinement phase.

In the present case, we firstly give n_b , then solve for $w(\rho, n_b)$. And by using this solution w , we obtain the chemical potential $\mu = A_0(\infty)$ from the following formula

$$\tilde{\mu} = \int_{\rho_m}^{\infty} d\rho F_{\rho t} = A_0(\infty) - A_0(\rho_m) = \mu - A_0(\rho_m), \quad (31)$$

where ρ_m is the minimum value of ρ and we remind $\tilde{F}_{\rho t} = 2\pi\alpha' F_{\rho t}$.

In order to make clear our viewpoint, we firstly restrict the analysis to the case of large m_q where $w(\rho, n_b = 0)$ is the Minkowski embedding solution. For $n_b = 0$, $\tilde{F}_{\rho t} = 0$, then the Minkowski embedding is physical. This does not mean $\mu = 0$, but it implies $A_0 = \mu$ where μ is constant. In this case, $\tilde{\mu} = 0$ and $A_0(\rho_m) = A_0(0) = \mu$. So we can consider two types of solutions for A_0 depending on the solution of w , (A) Minkowski type with $n_b = 0$ and (B) black hole type with $n_b \neq 0$, in the case of large m_q as given below.

For the BH embedding solutions with $n_b \neq 0$, the infrared boundary value $A_0(\rho_m)$ is estimated by using Eqs. (17) and (26). For enough small n_b , ρ_m is very small and the solution w very rapidly increases from the point $\rho = \rho_m$. It soon arrives at the value near the maximum value $w(\infty)$ at about $\rho = \rho_m + \epsilon$, $\epsilon \ll 1$. In other words, w is approximated by a step function then w' is done by the δ function. As a result for $1 \gg n_b \gg \rho_m^3$, we obtain

$$\int_{\rho_m}^{\infty} d\rho F_{\rho t} \sim \int_{\rho_m}^{\rho_m + \epsilon} d\rho A e^{\Phi/2} w' = \int_{r_0}^{w(\rho_m + \epsilon)} dr A e^{\Phi/2}, \quad (32)$$

for dS_4 with $q = 0$, and

$$\int_{\rho_m}^{\infty} d\rho F_{\rho t, T} \sim \int_{\rho_m}^{\rho_m + \epsilon} d\rho f \frac{r}{U} e^{\Phi/2} w' = \int_{r_T}^{w(\rho_m + \epsilon)} dr e^{\Phi/2}, \quad (33)$$

for finite temperature for any q . Here we notice the dilaton denoted by Φ in the above two equations are different from each other. Their explicit forms are given in the section two. For the case of dS_4 , however, the BH embedding is obtained only for $q = 0$ since $F_{\rho t}$ diverges at ρ_m for $q \neq 0$. Meanwhile such a situation can not be seen for the finite temperature phase, so we can consider any value of q in this case. On these points, we discuss again in the below.

In the limit of $\rho_m = 0$, the most right hand side of (32) and (33) are approximated as

$$\int_{r_0}^{w(0)} dr A e^{\Phi/2} = \tilde{m}_q, \quad \int_{r_T}^{w(0)} dr e^{\Phi/2} = \tilde{m}_{q, T}. \quad (34)$$

They are equivalent to the effective quark mass given in [22] and [20] defined from the Wilson line in dS_4 and at finite temperature respectively for the Minkowski embedding

case with $F_{\rho t} = 0$.[†] In both cases, the quarks are deconfined and moves in the corresponding gauge field medium with this effective mass. On the other hand, the left hand side of (32) and (33), by its definition, is equivalent to $\mu - A_0(\rho_m)$ at $\rho_m \rightarrow 0$. If we consider the chemical potential should be the effective quark mass in the limit $n_b \rightarrow 0$, then we should take as

$$A_0(\rho_m) = 0, \quad (35)$$

and we find in the limit of $n_b \rightarrow 0$

$$\tilde{\mu}(n_b \rightarrow 0) = \mu(n_b \rightarrow 0) \equiv \mu_0 = \tilde{m}_q \quad \text{or} \quad \tilde{m}_{q, T}. \quad (36)$$

We must notice here that this analysis is performed for $n_b \neq 0$ even if the limit of $n_b \rightarrow 0$ is taken, and the embedding profile is assumed to be the BH type.

The important point to be noticed here is that the value of μ_0 is finite. Then there is a region, $0 < \mu < \mu_0$ and $n_b = 0$, for finite λ or T in both cases. In these regions, the Minkowski embedding is allowed, then we can consider as mentioned above the following phase

$$\begin{aligned} \text{Phase (A)} \quad n_b = 0 : \quad & \{\rho_m = 0, \quad A_0(0) = \mu\} \\ & A_0 = \mu = \text{const.} \end{aligned}$$

On the other hand for $\mu_0 < \mu$, the following BH embedding phase (Phase (B)) is realized,

$$\begin{aligned} \text{Phase (B)} \quad n_b > 0 : \quad & \{\rho_m > 0, \quad A_0(\rho_m) = 0\} \\ & A_0(\rho) = \mu - \frac{n_b}{2\rho^2} + \cdots \quad \text{at large } \rho. \end{aligned}$$

And the critical curves separating (A) and (B) are given by $\tilde{m}_q(\lambda)$ and $\tilde{m}_{q, T}(T)$ for each model.

For this definition of two phases, we show the phase diagrams in the next section including numerical analyses. We should notice here that, in this phase transition, from phase (B) to (A), A_0 jumps from $A_0(\rho_m \neq 0) = 0$ to $A_0(0) = \mu_0$ at the limit of $\rho_m = 0$. Simultaneously, the embedding profile w changes from the BH to the Minkowski type. Actually from the left equations of the above approximate formula (32) and (33), we find $A'_0 \sim w'$, then we can understand the jump of A_0 corresponds to the jump of w near the horizon.

5 Phase transition at high temperature phase

Here we firstly solve the embedding equations by introducing the chemical potential and noticing that any Minkowski embedding solution is unphysical for $n_b \neq 0$. As mentioned above, the Minkowski embedding is considered only for $n_b = 0$. In this case,

[†] The authors of [14] discuss on this point related to the string energy from a similar viewpoint.

$\tilde{F}_{\rho t} = 0$ at any point, then we find physical Minkowski embeddings but there is no dynamical role of chemical potential since it vanishes completely from the action.

As stated above, two types of embeddings, (A) Minkowski and (B) BH embeddings, are possible, and they are discriminated by the form of A_0 and w . And the critical curve separating the two regions is given by $\mu_0 = \tilde{m}_q$. Before giving this critical curve, we discuss another transition which has been observed in [13, 14].

(i) Phase transition within (B) at small μ :

In the high temperature model, the topology changing phase transition is observed for $q = 0$ and $n_b = 0$. This transition is characterized by the jump of $w(\rho_m)$, where ρ_m represents the minimum point of ρ . In this sense, this transition is the first order since the D7 energy jumps at this point due to the different configuration of w . This kind of transition is also observed for finite n_b between the same BH embedding. So this is not the topology changing phase transition in the case of $n_b \neq 0$.

At the critical temperature T_1 , we find three different solutions at very small n_b for the same quark mass, $m_q = w(\infty)$. They are shown in the Fig. 1. This is firstly observed in [13]. In [13], w is assigned as the radial coordinate in X^8 - X^9 plane, but this is not important. We set as $w = X^8$, and we obtain almost the same result with [13].

The solutions are separated to two categories whether the limit value $\mu|_{n_b=0} (\equiv \mu_0)$ is finite (sol. (a)) or zero (sol. (b)). We notice here that the two solutions for the case of solution (b) are overlapped in the figure due to too small n_b . Then, this gives a gap of $\mu_0(T)$ at this temperature T_1 as seen in the phase diagram in μ - T plane given in the Fig. 3. The solution (a) provides finite μ_0 , and we observe

$$\frac{\partial \mu}{\partial n_b} < 0,$$

for this solution. This fact implies that this solution is unstable as pointed out in [14]. And this is seen for the very small μ region of $0 < \mu < \mu_1$, where μ_1 depends on m_q and is very small. As shown below, in the dS_4 model, there is no such a transition point at finite n_b so the gap is not seen. These points are assured in the phase diagrams given below.

Finally, we comment on the relation between the transition discussed here in terms of the Fig.1 and the one observed in the case of $n_b = 0$. The solution (a) in the Fig.1 transits to the Minkowski solution and the solution (b) remains as BH solution in the limit of $n_b \rightarrow 0$. This situation of the two BH configurations is schematically depicted in the Fig.4. In this sense, the transition considered here can be regarded as the limiting case of the topology changing phase transition which is seen at $n_b = 0$.

(ii) Phase transition from (B) to (A):

We notice that $\mu_0(T)$ is zero for $T > T_1$ and it becomes finite in the low temperature side $T < T_1$. (See Fig. 3.) The situation is here considered from the viewpoint of grand canonical ensemble, where the chemical potential μ is regarded as a parameter of the theory. In this sense, in the region of $0 \leq \mu < \mu_0$, the solutions (A) (Minkowski embedding) with $n_b = 0$ are found. When μ exceeds μ_0 , μ becomes active and could

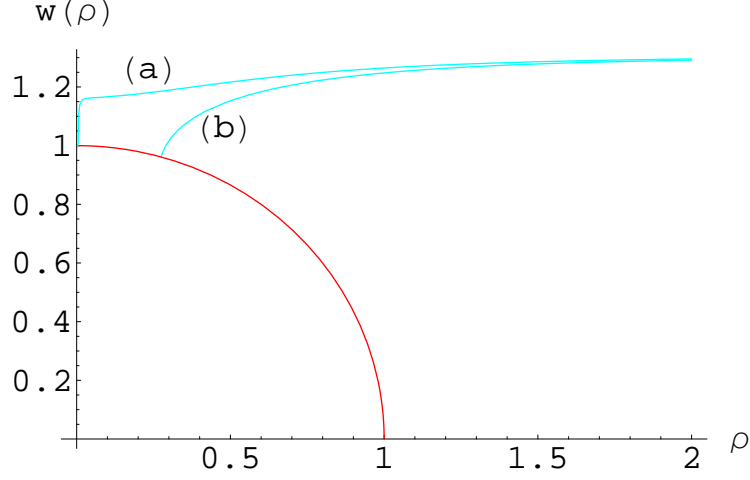


Fig. 1: Two embedding solutions for $q = 0$, $m_q = 1.309$, $n_b = 10^{-5}$ at the critical temperature $T_1 = 0.45$.

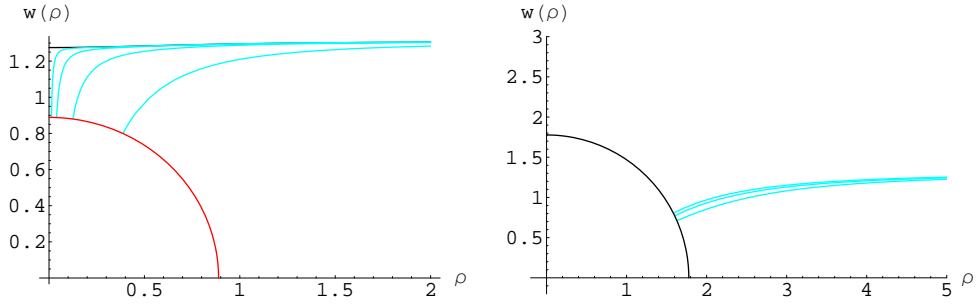


Fig. 2: Embedding solutions near the transition point for $q = 0$, $m_q = 1.30916$. The left are for $T = 0.4$, and $n_b = 0, 0.000187, 0.00184, 0.0181, 0.217$ from the above. The right is for $T = 0.8$, and $n_b = 0, 5, 10$.

generate a finite $n_b(\propto \langle \Psi^\dagger \Psi \rangle)$ to form the solution (B) (BH embedding). Then, for $\mu_0 < \mu$, the brane profile is changed to the BH embedding solution when its D7 energy density becomes smaller than the one of the Minkowski embedding with the same μ but with $n_b = 0$. We show in the below that this phenomenon really occurs at $\mu = \mu_0(T)$.

This implies a topology change of the D7 embedded configuration in the bulk 10 dimension, and, on the other hand, it is observed as the generation of $\langle \Psi^\dagger \Psi \rangle$ in the 4d gauge theory. The critical point $\mu_0(T)$ is determined for fixed $m_q = w(\infty)$ and T by taking the limit of $n_b = 0$ from the side of $n_b \neq 0$ (in the phase (B)).

The typical solutions of $w(\rho)$ near this transition point for $(T, m_q) = (0.4, 1.3)$ are shown for various μ in the left of the Fig. 2, where one Minkowski solution with $n_b = 0$ and four BH solutions with finite values n_b are shown. For the latter BH solutions, we obtain the corresponding values of $\mu(> \mu_0)$, where μ_0 at this temperature ($T = 0.4$) is read from the Fig. 3 as about $\mu_o = 0.14$. We notice that we have also the Minkowski solution for those value of μ with $n_b = 0$. Therefore, we must compare the D7 energy of each BH embedding solution and the corresponding Minkowski solution at the same value of μ in order to see which solution has lower energy. The D7 energy E_{D7}^T is defined as

$$S_{D7}^T = -\tau_7 \int d^7 \xi \epsilon_3^{1/2} E^T, \quad (37)$$

then for the Minkowski embedding of $q = 0$ we obtain

$$E_M^T = \int_0^\infty d\rho \rho^3 \left(\frac{r}{U} \right)^4 f e^\Phi \sqrt{1 + (w')^2}, \quad (38)$$

and for BH solution

$$E_{BH}^T = \int_{\rho_m}^\infty d\rho \rho^3 \left(\frac{r}{U} \right)^4 f e^\Phi \sqrt{(1 + (w')^2)(1 - G_T)}, \quad (39)$$

where G_T is given in (20). Each energy diverges, so we calculate the difference of E_M^T and E_{BH}^T ,

$$\Delta E^T = E_M^T - E_{BH}^T$$

to cancel the divergence coming from large ρ integration. At large ρ , the solutions of Minkowski and BH embeddings approaches to the same function, then this method of the regularization works well. And we could see $E_M^T > E_{BH}^T$ for any case, then the transition occurs at μ_0 between the Minkowski solution and the one of the BH with the smallest n_b . In other words, the curve $\mu_0(T)$ obtained in this way gives the critical curve which separates two phases (A) and (B), which are defined above.

In order to assure the statement, $\mu_0(T) = 0$ for $T > T_1$, given above, we also studied at $T = 0.8$ where all the solutions are BH type even if $n_b = 0$. The solutions are shown in the right hand side of Fig. 2. In this case, we could see that E_{BH}^T decreases with increasing n_b . So no Minkowski embedding appears as a stable state.

The resultant curve of $\mu_0(T)$ is shown in the Fig. 3. Near $T = T_1 \sim 0.45$, a small flat region is seen. This point is the transition point shown in (i) above. The

value of μ jumps from 0 to a small but finite μ_1 . Then, for $\mu_1 < \mu$, $\mu_0(T)$ increases monotonically with decreasing T , and arrives at $m_q = w(\infty)$ in the limit of $T = 0$. However, this diagram has its meaning only for $T > T_c$, where T_c denotes the deconfinement/confinement transition temperature since the present model describes only the deconfinement phase except for $T = 0$ and $q > 0$. While we do not discuss on this phase transition here,

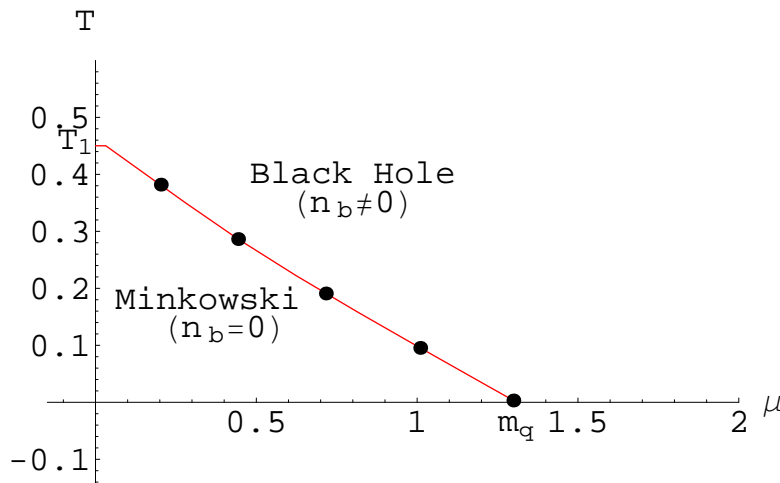


Fig. 3: Phase diagram in T - μ plane for $R=1, m_q = 1.30916$. The dots represent the effective quark mass \tilde{m}_q given by the last equation (34).

it is shown up to $T = 0$, from theoretical viewpoint.

The values of the effective quark mass \tilde{m}_q given by the last equations of (34) are shown in the Fig. 3 by several dots, and we can see they precisely coincide with $\mu_0(T)$ which are obtained as above.

At low temperature $T < T_c$, we should consider a model in the confinement phase. In our model, it is realized for $T = 0$ and $q > 0$. In this case, we know that \tilde{m}_q diverges [10] and this means that we can not observe an isolated quark. In other words, the theory is in the confinement phase. But we always find a finite \tilde{m}_q for $T > 0$ even if $q > 0$ in the present model, so our model describes the deconfinement phase so far as T is finite. Then $\mu_0(0)$ is infinite for $q > 0$, but the behavior of $\mu_0(T)$ for $T > 0$ is qualitatively similar to the one of $q = 0$. Then, the qualitative property is well described by $\mu_0(T)$ of $q = 0$. An important point is that the chemical potential is inactive for $\mu < \mu_0$, and there is a threshold where it becomes dynamical and to generate non-zero $n_b \propto \langle \Psi^\dagger \Psi \rangle$. This result implies that the chemical potential becomes active when it exceeds the effective quark mass since the gauge theory is in the quark deconfinement phase and there is no bound state of quarks. In order to make clear this point we consider the relation with a possible bound state.

As another possibility, we can consider μ_0 as the threshold of some bound state of quarks. In the present case, it would be the baryon with the mass M_B , then we will have $\mu_0 = M_B/N_c$ [19]. In order to prove this conjecture, we must include the baryon

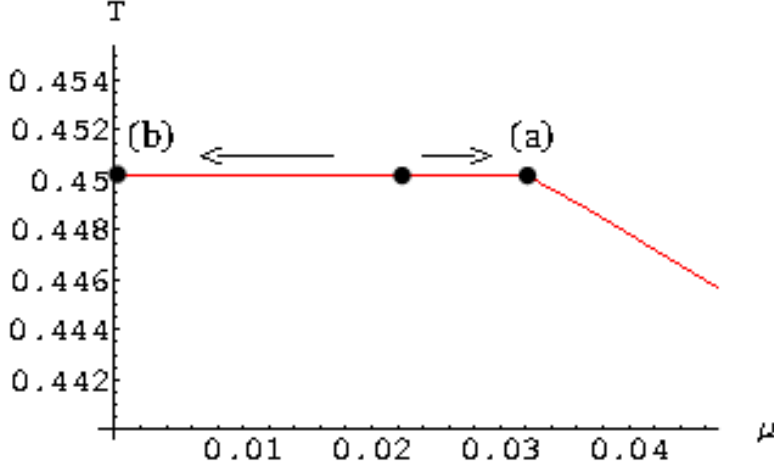


Fig. 4: The extended small μ region of the Fig. 3 is shown. At first, the two BH embeddings appear on the horizontal line at $(T, \mu, n_b) = (0.450158, 0.0223708, 0.00699)$ in a degenerated form. Then for decreasing n_b , they run on the line in the opposite direction shown by the arrows and approach to the point (a) and (b), which corresponds to the two BH configurations shown in the Fig.1.

in the model by for example introducing the D5 brane. In this paper, this point is an open problem and we will examine it in the near future.

In [19], the critical value of the isospin chemical potential is shown to be equal to the half of the meson mass in the case of quenched approximation. So this point could be examined by extending the model to non-Abelian DBI action [23]. In a related direction, an interesting analysis in the Higgs branch has been seen [24].

6 Phase transition in dS_4

In this case, the bulk configuration has a horizon at $r = r_0$ which determines the 4d cosmological constant λ as $\lambda = 4r_0^2/R^4$. Then the both the Minkowski BH embeddings are seen, and we can expect the change of profiles as in the finite temperature case. Actually, a kind of phase transition is seen in the case without the chemical potential but with gauge field condensate [22]. Here we are solving the embedding equations by introducing the chemical potential.

In this case, we should notice the following points. (i) For the Minkowski embedding, $\tilde{F}_{\rho t}$ should vanish at the end point $\rho = 0$. (ii) As for the BH embeddings, the end point of the brane is at the horizon $r = r_0$. Near this point, we find $e^\Phi \propto q(1 - r_0/r)^{-3}$ then

$$\tilde{F}_{\rho t} \propto q^{1/2} n_b (1 - r_0/r)^{-1/2}$$

which implies $\tilde{F}_{\rho t} = \infty$ at $r = r_0$. In this case, we can not find any black hole

embeddings. As a result, the black hole embedding solution can not be obtained for finite q (finite gauge condensate $\langle F_{\mu\nu}^2 \rangle$) and $n_b(\neq 0)$.

Meanwhile, for $q = 0$ and $n_b \neq 0$,

$$e^\Phi = 1, \quad \tilde{F}_{\rho t} \propto (1 - r_0/r) \rightarrow 0$$

in the limit $r = r_0$, and this is independent of n_b . Then, in the case of $q = 0$, we find $\tilde{F}_{\rho t} = 0$ on the horizon. It shows that the black hole embedding is physical in this case, and we actually could find such solutions.

Therefore, we restrict hereafter to the case of $q = 0$ for dS_4 model in order to allow the physical embeddings of $n_b \neq 0$ case as the black hole embedding, since it is the only possible embedding in the case of $n_b \neq 0$. On the other hand, as mentioned above, the Minkowski embedding is allowed only for $n_b = 0$ (phase (A)). But, we remind that this does not necessarily mean $A_0 = 0$. For this embedding, we can consider a finite chemical potential, $A_0 = \mu$ where μ is finite but $n_b = 0$. In this case, $\tilde{F}_{\rho t} = 0$ at any point, then we find physical Minkowski embeddings but there is no dynamical role of chemical potential since it vanishes completely from the action. As in the above case, the phase (A) is realized for $0 < \mu < \mu_0(\lambda)$, and $\mu_0(\lambda)$ represent the critical curve of the phase transition from phase (A) to (B). And this curve is obtained according to the method given for the finite temperature case.

Fixing m_q and λ , the typical solutions of $w(\rho)$ for $q = 0$ near this transition point are shown in the Fig. 5.

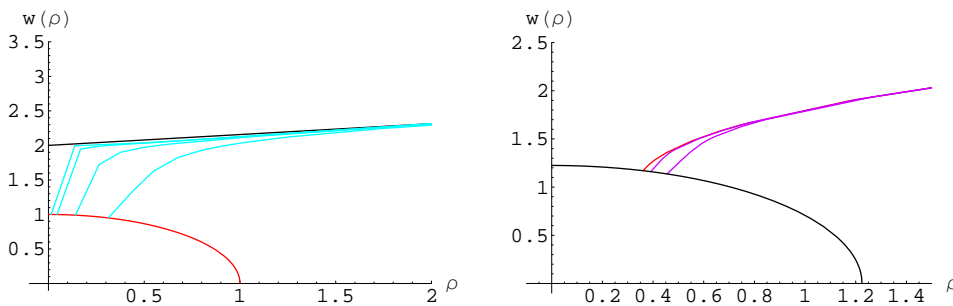


Fig. 5: Embedding solutions near the transition point for $q = 0$, $m_q = 2.94966$. The left are for $\lambda = 4$, and $n_b = 0, 0.000124, 0.00124, 0.0133$ and 0.0971 from the above. The right is for $\lambda = 6$, and $n_b = 0, 0.001, 0.01$.

In the left of the Fig. 5, four BH solutions have finite values of n_b as shown in the figure caption. They have also the corresponding values of $\mu > \mu_0$, where μ_0 is shown in the Fig. 6. For those values of μ , we have also the Minkowski solution given in the Fig. 5 with $A_0 = \mu$. Then, we compare the D7 energy of each BH embedding solution and the corresponding Minkowski solution at the same value of μ in order to see which solution has lower energy. The D7 energy E_{D7} for $q = 0$ which is defined as [22]

$$S_{D7} = -\tau_7 \int d^7\xi \sqrt{\epsilon_3} a(t)^3 E_{dS_4} \quad (40)$$

then for Minkowski embedding solution,

$$E_{\text{dS}_4}^M = \int_0^\infty d\rho \rho^3 A^4 e^\Phi \sqrt{1 + (w')^2} , \quad (41)$$

and for BH solution,

$$E_{\text{dS}_4}^{BH} = \int_{\rho_m}^\infty d\rho \rho^3 A^4 e^\Phi \sqrt{(1 + (w')^2)(1 - G)} , \quad (42)$$

where G is given by (28). But as in the case of finite temperature, E_{dS_4} is also divergent. Although a way to regularization is given in [22], we estimate the difference,

$$\Delta E \equiv E_{\text{dS}_4}^M - E_{\text{dS}_4}^{BH}$$

as in the previous section at the same quark mass m_q and λ since ΔE should be finite.

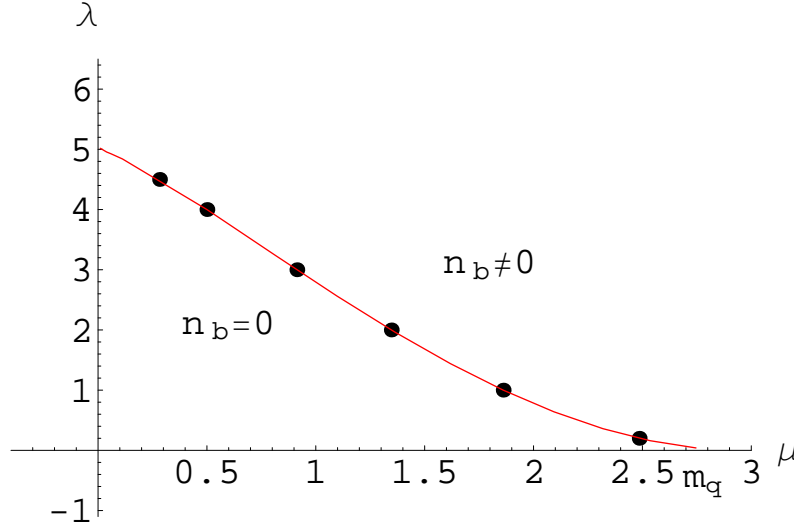


Fig. 6: The value of μ_0 for $m_q=2.94966$ and $R=1$. The dots represent the effective quark mass calculated by (34) for $m_q = 2.94966$.

According to this procedure, we find that all the above black hole solutions give lower energy than the one of the Minkowski embedding, and the energy decreases with increasing n_b . Then the critical value μ_0 at this λ is obtained as the limiting value of μ obtained by approaching to $n_b = 0$ from BH embedding side. For $\lambda > 5$, this limiting value is obtained as zero. Then the all the embeddings are the BH type even if $n_b = 0$ as shown by the right one of the Fig. 5. Performing this procedure for other points of $\lambda < 5$, we find the critical curve in the λ - μ plane. The results are shown in the Fig. 6. As in the high temperature case, we can see the equivalence of μ_0 and the effective quark mass \tilde{m}_q obtained from the formula (32). Then this result is interpreted as the common property of the deconfinement phase gauge theory with quarks

7 Summary

Here the role of the chemical potential of the quark is studied in terms of the holographic gauge theory at finite temperature and in dS_4 . Quarks and the chemical potential are introduced by embedding the probe D7 branes in the bulk configurations corresponding to the considering gauge theories. In both bulk configurations, there is a horizon. Then two embedding forms, Minkowski and black hole, are possible. The chemical potential μ and the number density n_b are introduced through the Wick rotated time component of $U(1)$ vector field, A_0 , on the D7 brane.

In order to embed the D7 brane, the profile function $w(\rho)$ and $A_0(\rho)$ are solved by the variational principle of the D7 action. The equations of motion are solved firstly for w by giving n_b , then we obtain μ . This method correspond to study the system in the micro-canonical ensemble. In this approach, we find a finite $\mu(= \mu_0)$ in the limit of $n_b = 0$. This implies the existence of a parameter region of $0 < \mu < \mu_0$ and $n_b = 0$, where the D7 brane can not be embedded by the BH embedding. So we need Minkowski embedding in this region in order to introduce quarks.

Actually, in this region the Minkowski embedding is physical since $n_b = 0$ and we obtain the electric potential $A_0 = \mu$, then A_0 is a constant with respect to ρ . Thus, in this case, we can study the system from the viewpoint of grand canonical ensemble, where μ is the given parameter. So, by varying the chemical potential from zero to large μ , we find that μ is inactive below μ_0 and $\langle \Psi^\dagger \Psi \rangle$ appears for $\mu_0 < \mu$ and increases with μ .

At the same time with the appearance of $\langle \Psi^\dagger \Psi \rangle$, the profile of the D7 embedding is changed to the BH form. This is assured by comparing the D7 energies of Minkowski and BH embeddings with the same μ and m_q at fixed T (or λ). The energy of the BH embedding with finite n_b is always smaller than the one of Minkowski embedding. This is therefore a kind of phase transition with the order parameter $\langle \Psi^\dagger \Psi \rangle$. In the bulk, this is seen as the topology changing of the D7 embedded configuration. We assured this phase transition in the two different theories which are in the quark deconfinement phase. In both cases, the critical point μ_0 is identified with the effective quark mass \tilde{m}_q . Thus, this result is consistent with the picture that the quark in the deconfinement phase moves freely with this effective mass.

Then the transition stated above is common to the two deconfinement model. But we observe some differences between the two models. In both cases, we introduce a parameter q which is representing the gauge condensate $\langle F_{\mu\nu}^2 \rangle$. And this parameter is responsible to the quark confinement. Actually, for $\lambda = 0$ and $T = 0$, the two theory are equivalent and the theory is in the confinement phase. On the other hand, for $T > 0$ or $\lambda > 0$, both theories changes to the deconfinement phase. However, when n_b is added, while the BH embedding is possible in the finite temperature theory for any value of $q > 0$, it is forbidden for dS_4 .

Another difference is a phase transition which is seen only for the finite temperature case near very small μ , where a embedding configuration change is seen between the same BH type configurations. This is not seen in the case of dS_4 .

The interesting case would be the model in the confining phase, where the effective quark mass diverges, then μ_0 diverges. In other words, we can not find finite $\langle \Psi^\dagger \Psi \rangle$ at any μ . The only case, where $\langle \Psi^\dagger \Psi \rangle$ is seen in the confinement phase would be in a model in which the baryon is introduced. In this case, we will find the transition point at $\mu_0 = M_B/N_c$ through the same analysis, where M_B is the lowest baryon mass. On this point we will discuss in the future.

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